1 Abstract

Projectiles follow parabolic paths and planets move in elliptical orbits. Circles, hyperbolas, parabolas and ellipses are curves that are so abundant in nature, engineering, and art that we cannot help but notice them. Each of these curves is an example of a conic. In 1848, the mathematician Jacob Steiner posed a famous question: "How many conics are tangent to five fixed conics?" Steiner claimed to have solved the problem and he gave the answer 7776. This solution was accepted as valid for sixteen years. When the problem was revisited in 1864, the mathematician Michel Chasles realized that Steiner had miscounted the true number of conics that satisfied the conditions. Not all conics are smooth plane curves. Singular conics are curves whose defining polynomials are reducible to the product of two linear factors. These conics can be represented as either a pair of crossed lines or a line of multiplicity two. Steiner failed to account for the degenerate conics that can be represented as a double line. He fell victim to what algebraic geometers call excess intersection. This Trident project is centered on understanding how excess intersection affects problems of enumeration involving plane conics. Research was focused on finding the solutions to twenty-one variations of Steiner's problem. These problems were solved by examining the blowup of the space of conics along the set of double lines and executing computations in what is known as the Chow Ring. These methods provide not only tangible numerical results but help to illuminate the rich underlying geometry of these fundamental problems.

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13. ABSTRACT: Projectiles follow parabolic paths and planets move in elliptical orbits. Circles, hyperbolas, parabolas and ellipses are curves that are so abundant in nature, engineering, and art that we cannot help but notice them. Each of these curves is an example of a conic. In 1848, the mathematician Jacob Steiner posed a famous question: "How many conics are tangent to five fixed conics?" Steiner claimed to have solved the problem and he gave the answer 7776. This solution was accepted as valid for sixteen years. When the problem was revisited in 1864, the mathematician Michel Chasles realized that Steiner had miscounted the true number of conics that satisfied the conditions. Not all conics are smooth plane curves. Singular conics are curves whose defining polynomials are reducible to the product of two linear factors. These conics can be represented as either a pair of crossed lines or a line of multiplicity two. Steiner failed to account for the degenerate conics that can be represented as a double line. He fell victim to what algebraic geometers call excess intersection. This Trident project is centered on understanding how excess intersection affects problems of enumeration involving plane conics. Research was focused on finding the solutions to twenty-one variations of Steiner's problem. These problems were solved by examining the blowup of the space of conics along the set of double lines and executing computations in what is known as the Chow Ring. These methods provide not only tangible numerical results but help to illuminate the rich underlying geometry of these fundamental problems.

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Contents

3 Introduction

3.1 Mathematical Introduction

A conic is the simplest geometric object that possesses curvature. Circles, hyperbolas, parabolas and ellipses are conics that are so abundant in nature and engineering that we can not help but notice them. Projectiles follow parabolic paths and planets move in elliptical orbits. For our purposes, a conic is a degree-two curve. We will examine various examples of conics in section 4.2.

The defining equation for a projective conic is $ax^2+by^2+cz^2+dxy+exz+fyz=0$. That is, a curve in the projective plane is a conic if it is of this form. By varying the six coefficients in this equation we shift from one conic to another. In fact, if we keep this defining equation consistent then the variables x, y and z become unimportant and we can drop them. Every conic can be described with a coordinate of the form $[a:b:c:d:e:f]$. This coordinate is a point in five dimensional projective space which we denote by the symbol $\mathbb{P}^5$. In projective space six coordinates define a five dimensional point. This is a powerful observation, a conic in the plane can be represented as a simple point in $\mathbb{P}^5$. We can answer many questions involving conics by working only with points in $\mathbb{P}^5$. Also, surfaces in $\mathbb{P}^5$ correspond to large sets of conics in the plane. For example, the set of all conics that pass through a fixed point in the projective plane form a four dimensional hyperplane in $\mathbb{P}^5$.

The most famous example of an enumerative problem in algebraic geometry is *Steiner's Problem*. Steiner asked the following question, "How many conics are tangent to five fixed conics?"[FS] This problem has an interesting history. Steiner claimed to have solved the problem and gave an answer of 7776. For a large portion of the nineteenth century, his answer was accepted as valid. Steiner went wrong when he incorrectly applied a powerful result known as Bezout's theorem. The discrepancy between the actual number of conics satisfying the given conditions and the number that Bezout's Theorem returned, was a result of what is know as *excess intersection*. Not all conics are smooth curves like parabolas or circles. There is a type of conic called a *singular* conic that looks very different from the

smooth curves that we are used to thinking about. Singular conics can be visualized as a pair of lines in the plane. More often than not these lines simply cross at a single point, but sometimes the two lines come together and lie on top of each other. This special type of conic is known as a double line conic. It not only looks very different from most degree-two curves, it creates difficulties in our ability to solve enumerative problems. Two conics are tangent to each other if they intersect in fewer than four distinct points. When given a smooth conic such as a circle, and a double line, the two will *always* intersect in fewer then four distinct points. Thus every double line conic is tangent to every other possible conic. Since there are infinitely many double line conics, we can see how they might interfere with our counting when solving enumerative problems. Before we can set out to actually answer questions such as the one Steiner posed, we need to develop a method of describing the set of all double line conics. From there we will be better able to avoid these particular algebraic varieties when solving problems.

Every conic in the plane corresponds to a point in $\mathbb{P}^5$ and double lines are no exception. The set of all double line conics form a two-dimensional surface in the parameter space of all possible conics. We refer to this large group of double lines as the *Veronese surface* and it plays a crucial role in understanding enumerative problems involving conics. We mentioned earlier that the set of all conics passing through a point form a four dimensional plane in $\mathbb{P}^5$. Similarly, the set of all conics tangent to a fixed line form a higher dimensional surface in $\mathbb{P}^5$. However, since every double line is tangent to a fixed line, this surface in $\mathbb{P}^5$ completely contains the Veronese surface. So when we ask the question, "How many conics are tangent to five fixed lines?" we can't try to solve the problem by simply intersecting the five corresponding surfaces in $\mathbb{P}^5$.

We resolve this problem by what is known as *blowing up* the space. $\mathbb{P}^5$ is stretched along the set of points that correspond to double line conics in the plane. The part of $\mathbb{P}^5$ that lies outside of the Veronese surface is left intact after this stretching. Our new space that is formed by blowing up $\mathbb{P}^5$ is much larger. By stretching out the space of all conics along the Veronese surface, we are better able to examine the surfaces that initially

caused problems. Four-dimensional surfaces that completely contained the Veronese in $\mathbb{P}^5$ no longer do so in the blowup of the space. In addition, the troublesome double line conics no longer prevent us from correctly intersecting surfaces in $\mathbb{P}^5$. In order to work with our special algebraic varieties inside of the blowup of $\mathbb{P}^5$ we introduce what is known as the Chow ring. The Chow ring provides us with a set of algebraic operations that give insight into the underlying geometry. These techniques allow us to correctly answer a wide variety of enumerative problems, including Steiner's problem.

For any projective space of dimension n the set of $n-1$ dimensional linear subvarieties are parameterized by the points in $\mathbb{P}^n$. In the paper we will examine the deep relationship between the blowup of $\mathbb{P}^5$ along the Veronese and this property of duality that is innate to projective space.

With the help of modern algebraic geometry and the concept of the moduli space we can revisit some problems in geometry that originated in ancient Greece, in particular the *Appolonius Circle Problem.* Appolonius of Perga posed the question, "Is it possible to construct all circles that are tangent to three given circles?" Although this problem was solved by the French geometer Francois Viete during the Enlightenment, we can verify his results quite elegantly with some modern methods.

3.2 Overview of the Report

In Chapter 4 we formally introduce the reader to projective space and projective varieties. We define the complex projective conic and establish the correspondence between conics in the projective plane and points in five dimensional projective space. We also discuss the concept of duality in projective space and define the notion of degree. The last part of the chapter is dedicated to familiarizing the reader with the difficulties that arise when applying naive counting methods to enumerative problems. In Chapter 5 we discuss important subvarieties that lie within the moduli space of conics. We introduce the Veronese surface as a space of double line conics and describe how it affects our counting methods. We also examine the Segre variety, the space of crossed line conics.

Chapter 6 is dedicated to developing a method to solve the variations of the Steiner problem. We discuss the blowup of the space of all conics along the Veronese surface and the relationship between the blowup and the concept of duality. In Chapter 7 we define the Chow ring and the concept of rational equivalence. We derive a powerful identity that relates important surfaces in $\mathbb{P}^5$.

Appolonius' circle problem is a famous problem in classical geometry. In Chapter 7 we revisit this classical problem with the aid of techniques from modern algebraic geometry.

4 Projective Varieties

4.1 Complex Projective Space

The world that we live in is three dimensional. Every point in space can be pinpointed with three spacial coordinates. To a mathematician, we live in what is known as three dimensional Euclidean space. Objects in this space look and intersect one another just as they would in nature. Another attribute that is commonly associated with Euclidean space is the coordinate system. Once we establish a consistent way of measuring distance and an agreed upon origin, the location of every point within the space can be charted. Affine space can be thought of as our familiar Euclidean space without the luxury of a fixed origin. A plane in three dimensional Affine space looks just like a flat region of the Earth, but if you were to stand on that plane you would not be able to identify your location. As with Euclidean space, if you were to walk on top of a plane in Affine space you would be able to walk forever in any direction. Everything would appear to be just boring open space that continues on as far as the eye can see.

Algebraic geometers frequently work with a more counterintuitive space. Projective space is what is known as a *compactification* of Affine space. We can visualize the compactification as a compression of this large infinite expanse of space into something more manageable. No matter how far you walk in Affine space you will never make any progress, since you are inside a void that continues on forever. But when you walk within projective space, you will

see yourself gradually approaching points that are infinitely far away. In fact, points that are infinitely far away are actual tangible points that can be reached in projective space.

Why would we want to study geometry in such a counterintuitive space? The answer is that when we consider more difficult problems, it makes work easier. For example, we know that two lines in a plane in Euclidean or Affine space will intersect each other in at most one point. The only time that the two lines will not intersect is when they are parallel. In projective space however, two parallel lines will actually meet each other at a point which is infinitely far away. This simplifies matters somewhat since we are now able to say that two distinct lines lying in the projective plane will always intersect at one point.

Definition 1. Complex projective space, denoted by $\mathbb{P}^n$, is the set of all one-dimensional complex subspaces of the complex vector space $\mathbb{C}^{n+1}$. That is, $\mathbb{P}^n$ is the set of all complex lines through the origin in $\mathbb{C}^{n+1}$.

Example 2. A line through the origin in $\mathbb{C}^3$ is a point in two-dimensional projective space. The set of all such lines forms $\mathbb{P}^2$, or the *projective plane*.

A good general reference for projective space is [SKKT] while [C] and [SC] treat the projective plane in great detail.

We can fix a reference plane in $\mathbb{C}^3$ that does not pass through the origin and identify each point on the the reference plane with a point in $\mathbb{P}^2$. Each point on this affine plane is identified with a line in $\mathbb{C}^3$ that passes through both the point and the origin. The only points in $\mathbb{P}^2$ not identified in this manner are the points corresponding to lines in $\mathbb{C}^3$ running through the origin, parallel to our reference plane. These points in $\mathbb{P}^2$ form a one-dimensional projective space and are referred to as the *line at infinity* [SKKT]. These points are added to our reference plane to give a natural compactification of two dimensional affine space.

We can generalize these ideas to all higher dimensional projective spaces and express the concepts formally with the following mapping

$$\mathbb{P}^n \;=\; \mathbb{C}^n \cup \mathbb{P}^{n-1}$$

$$[x_0 : x_1 : \cdots : x_n] \mapsto \begin{cases} (\frac{x_1}{x_0}, \ldots, \frac{x_n}{x_0}), & \text{for } x_0 \neq 0, \\ [x_1 : \cdots : x_n], & \text{for } x_0 = 0 \end{cases}$$

When x_0 is nonzero, the point in $\mathbb{P}^n$ is taken to a point in $\mathbb{C}^n$. When x_0 is equal to zero, then our map takes the corresponding point in $\mathbb{P}^n$ to a point on the $\mathbb{P}^{n-1}$. We refer to the points in $\mathbb{P}^n$ that lie in our reference plane $\mathbb{C}^n$ as points in the finite part of $\mathbb{P}^n$ and points on $\mathbb{P}^{n-1}$ as points at infinity.

When we choose a point p in n dimensional projective space $\mathbb{P}^n$, we are in fact dealing with an equivalence class of points in $\mathbb{C}^{n+1}$. There is a one to one correspondence between lines through the origin in $\mathbb{C}^{n+1}$ and points in $\mathbb{P}^n$. We can represent a point in projective space formally as

$$[(x_0 : x_1 : \cdots : x_n)] = \{(\lambda x_0, \lambda x_1, \ldots, \lambda x_n) \mid \lambda \in \mathbb{C}\}.$$

The bracketed coordinate notation on the left side of the above equality is known as the *homogeneous coordinate* of the representative point in projective space and is used to denote the class of equivalent points in affine space.

4.2 Projective Conics

Definition 3. A complex projective conic is the locus of roots $[x : y : z]$ of a degree-two homogeneous polynomial $F(x, y, z) : ax^2 + by^2 + cz^2 + dxy + exz + fyz = 0$ in $\mathbb{P}^2$ with coefficients $a, b, c, d, e, f \in \mathbb{C}$ where not all of the coefficients are zero [C],[SKKT],[SC],[W]. This locus of roots is often referred to as a *vanishing set* and is denoted $\mathbb{V}(F)$.

Our above definition asserts that a plane curve is a conic if and only if it satisfies the general equation. There are several types of planar varieties that satisfy this equation. Not all of them, however, have similar geometric interpretations. The next two definitions will break the set of all conics into two important classes: *singular* and *non-singular*.

Definition 4. A conic $\mathbb{V}(G) \subset \mathbb{P}^2$ is said to be *non-singular* or *smooth*, if its defining polynomial G cannot be represented as the product of two linear factors.

Example 5. The parabola $x^2 - yz = 0$ is a non-singular conic since the polynomial $(x^2 - yz)$ does not factor into two linear factors. The parabola is a canonical example of a smooth conic. It is smooth since it has a well defined tangent line at every point.

Definition 6. A conic $\mathbb{V}(F) \subset \mathbb{P}^2$ is said to be *singular* if its defining polynomial F can be reduced to the product of two linear factors. That is, $\mathbb{V}(F) = \mathbb{V}(G_1 G_2)$ where G_1 and G_2 are degree-one polynomials [FS].

A singular conic can be thought of as the set of points in the plane that lie on two crossed lines. This geometric interpretation fits nicely with the definition because when we speak about conics we are speaking about vanishing sets. Note that $G_1 G_2 = 0$ precisely when either G_1 is zero or G_2 is zero, so a point lies on the conic $\mathbb{V}(G_1 G_2)$ if and only if it lies one one of the lines $\mathbb{V}(G_1)$ and $\mathbb{V}(G_1)$.

Example 7. The conic $\mathbb{V}(20x^2 + 3y^2 + 19xy - 35xz - 7yz)$ is *singular* since the polynomial factors and we have $\mathbb{V}((4x + 3y - 7z)(5x + y))$. This variety can be viewed as the set of all points that lie on either the line $4x + 3y - 7z = 0$ or the line $5x + y = 0$, as in Figure 2.

Example 8. The conic $\mathbb{V}(9x^2 + 36xy + 36y^2)$ is *singular* since it is equivalent to the conic $\mathbb{V}((3x + 6y)^2)$. This is an example of a *double line conic*, a special type of singular conic. It can be thought of as two crossed lines that have come together as in Figure 1.

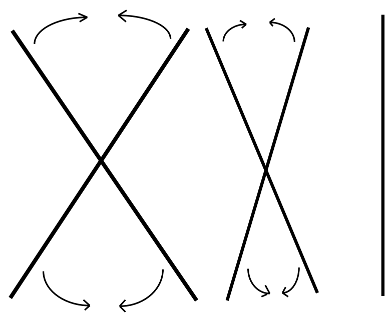

Figure 1: Rotating the crossed lines produces a double line conic

We see that in the case of degree two, there is only one type of degenerate curve, namely crossed line conics. This is not true of curves of a higher degree. Smooth conics are the

curves that mathematicians and scientists are often most interested in. These are the "natural" conics that exhibit curvature. The distinction between smooth and singular conics is important to understand since singular varieties pose many difficulties when struggling with problems involving the intersection of projective curves. We can see this when we consider the intersection of specific conics.

In a typical situation, two smooth conics will intersect each other in four distinct points. This can be seen by laying two real ellipses E_1 and E_2 on top of one another with their respective major axis at right angles. This situation can be expressed algebraically with the defining polynomials of the two ellipses. By parameterizing the curve E_2, the intersection of E_1 and E_2 can be represented by a degree four polynomial which is the restriction of E_1 to E_2. This polynomial will have four distinct roots. Each distinct root corresponds to an intersection of the algebraic varieties.

Two conics are said to be *tangent* to one another if they intersect in fewer then four distinct locations. If two smooth conics are tangent to one another, as in Figure 2, then this tangency can be seen algebraically by examining the restriction of one curve to another. The resulting polynomial will have a multiple root corresponding to the point of tangency.

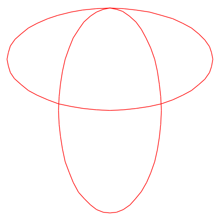

Fig. 2: Tangent Ellipses

This idea of tangency is quite clear when we are dealing only with smooth conics. It can be interpreted as two curves coming together and touching at a point. What if we consider a double line conic like the one introduced in Example 8? Every double line conic intersects a smooth conic in only two distinct places. Therefore, within the complex projective plane,

every double line conic is tangent to every smooth conic. This illustrates just how careful we must be when addressing problems involving the intersection of plane conics.

4.3 A Parameter Space for Conics

In the previous section we gave a formal definition for the projective conic that will help us to classify and work with individual plane curves. The definition also leads us in a natural way to a method by which we can examine large groups of conics.

We consider a conic $\mathbb{V}(ax^2+by^2+cz^2+dxy+exz+fyz) \subset \mathbb{P}^2$ with coefficients a,b,c,d,e,f and assume that not all of the coefficients are zero. If the order of variables in this general equation is kept consistent, then every possible degree 2 curve in the plane can be represented by six complex coefficients. We observe that this induces a one to one correspondence between the set of conics in $\mathbb{P}^2$ and the points $[a:b:c:d:e:f]$ in $\mathbb{P}^5$. For example, consider the conic $x^2-5y^2+3xz=0$. We can describe this conic with the point $[1:-5:0:0:3:0]$, a point in $\mathbb{P}^5$.

Our parameter space for plane conics can be constructed with the following mapping

$$\{Conics\, in\, \mathbb{P}^2\} \longrightarrow \mathbb{P}^5$$

$$\mathbb{V}(ax^2+by^2+cz^2+dxy+exz+fyz) \longmapsto [a:b:c:d:e:f].$$

This is a powerful observation, for it allows us to classify large families of plane conics by noting relationships between points in a higher dimensional projective space. An important property of $\mathbb{P}^5$ is that points are unaffected by scalar multiplication. That is, The point $[1:0:1:0:-1:0]$ is equal to the point $[2:0:2:0:-2:0]$. This makes sense since the the corresponding conics have equations $x^2+z^2-xz=0$ and $2x^2+2z^2-2xz=0$ but describe the same set of points in the plane.

An algebraic variety in $\mathbb{P}^5$ corresponds to a set of degree 2 curves in the complex projective plane. Each point on the variety corresponds to a conic in $\mathbb{P}^2$. We refer to $\mathbb{P}^5$ as the *moduli space* of conics embedded in $\mathbb{P}^2$. Lemma 10 gives insight into how we will use this moduli space to better understand enumerative problems involving conics.

Before we prove Lemma 10 we need to prove a very short but important result that we will make use of throughout the paper.

Lemma 9. *If a conic C contains three points that all lie on a line L, then it must have a linear factor.*

Proof. Assume that C is a smooth conic. We know that it intersects L in at least three points. But this is a contradiction to the fact that a line can only intersect a conic in at most two points. Thus C is a singular conic. □

Lemma 10. *Given an arrangement of n points in $\mathbb{P}^2$, the set of all conics passing through each of the fixed points forms a hyperplane condition in the moduli space of plane conics. Moreover, if no four of the points are collinear then these hyperplanes $H_1, H_2, ...H_n$ are linearly independent for $n \leq 5$.*

Proof. Begin by fixing a point $P_0 : [x_0 : y_0 : z_0]$ in $\mathbb{P}^2$. The condition that a conic pass through the point P_0 is given by the equation

$$a{x_0}^2 + b{y_0}^2 + c{z_0}^2 + dx_0y_0 + ex_0z_0 + fy_0z_0 = 0$$

or

$${x_0}^2a + {y_0}^2b + {z_0}^2c + x_0y_0d + x_0z_0e + y_0z_0f = 0.$$

We can see this is a codimension-one linear subvariety, or hyperplane in $\mathbb{P}^5$.

Two hyperplanes are linearly independent if their corresponding normal vectors are linearly independent. Consider three hyperplanes in $\mathbb{P}^5$ that are formed by the set of all conics passing through three points $P_0, P_1, P_2 \in \mathbb{P}^2$ and assume that the three hyperplane are linearly dependent. Therefore we know that one of the hyperplanes, say H_2, contains the intersection $H_0 \cap H_1$. The codimension-two intersection corresponds to the set of all conics that pass through the points P_0 and P_1. The fact that the third hyperplane contains this space implies that any conic passing through P_0 and P_1 must also pass through P_2. This is a contradiction: since if the three points are not all collinear, then we can choose a double line conic that passes through P_0 and P_1 but will not pass through P_2. On the other hand,

if the three points are all collinear then any non-singular conic that passes through P_0 and P_1 does not pass through P_2 or else it would be a product of linear factors. Thus any three points in $\mathbb{P}^2$ form three linearly independent hyperplanes in $\mathbb{P}^5$.

Let H_0, H_1, H_2 and H_3 be four hyperplanes in $\mathbb{P}^5$ formed by the set of conics passing through points $P_0, P_1, P_2, P_3 \in \mathbb{P}^2$ respectively. Assume that the hyperplanes are linearly dependent. Therefore we know that the fourth hyperplane, say H_3, contains the intersection of H_0, H_1 and H_2 since we showed that any three hyperplanes are linearly independent in $\mathbb{P}^5$. This implies that conics that pass through P_0, P_1 and P_2 must also pass through P_3. For the first case assume that P_0, P_1 and P_2 are not all collinear. Then there is a set of crossed line conics that pass through P_0, P_1 and P_2 but do not pass through P_3: namely a line passing through P_0 and P_1 and another line that passes through P_2 but does not pass through P_3. This is a contradiction to the assumption that the four hyperplanes are dependent. For the second case, assume that three of the fixed points are collinear. We can choose a conic that passes through the three collinear points but does not pass through the fourth point, namely a double line. For the third case assume that P_0, P_1, P_2 and P_3 are all collinear. The only conics that can pass through the first three points are conics that can be represented as the product of linear factors. In this case the set of conics that pass through the first three of the fixed collinear points must also pass through P_3. Thus we have established that four points in the plane that are not all collinear form four linearly independent hyperplanes in $\mathbb{P}^5$. And if the four points are collinear if and only if there is a dependence condition between the corresponding hyperplanes in $\mathbb{P}^5$.

Let H_0, H_1, H_2, H_3 and H_4 be five hyperplanes in $\mathbb{P}^5$ formed by the set of conics passing through points $P_0, P_1, P_2, P_3, P_4 \in \mathbb{P}^2$ respectively. Assume that no four of the points are collinear and that the hyperplanes are linearly dependent. The fifth hyperplane, H_4, therefore contains the intersection of the other four.

Assume that no three of the points P_0, P_1, P_2 and P_3 are collinear. Our linear dependence assumption implies that all conics that pass through these four points must pass through a fifth point P_4. This is not true since we can consider a crossed line conic that contains P_0

and P_1 on one line and P_2 and P_3 on the other. Such a conic probably does not contain P_4. If this conic does happen to contain P_4, then we can choose a new pair of crossed lines, say one line containing P_0 and P_2 and the other containing P_1 and P_3, and this conic will definitely not contain P_4 since no three of the points P_0, P_1, P_2 and P_3 are collinear.

If three of the five points are collinear then we have a one dimensional space of conics that pass through four of the points but do not contain the fifth. Namely, conics that are formed by a line that passes through the three collinear points and a $\mathbb{P}^1$ of lines that pass through the fourth point. We can thus choose one of these lines such that we now have a pair of crossed lines that do not contain our fifth point.

We have therefore established that by fixing $n \leq 5$ points in the plane such that no four of the points are collinear, we form n linearly independent hyperplanes in $\mathbb{P}^5$. □

We can make use of $\mathbb{P}^5$ and Lemma 10 to solve our first enumerative problem. The following result is known as the *Five Points Theorem* and is fundamental to our understanding of plane conics. It was known to the ancient Greeks that five points in the plane determine a conic. There are many classical constructions for producing the unique conic that passes through five fixed points. The proof of the following theorem shows the power of modern algebraic geometry and the concept of the moduli space.

Theorem 11. *Given any five points in $\mathbb{P}^2$ such that no four points are collinear, there exists a unique conic that contains them all.*

Proof. Consider the fixed point $[x_1 : y_1 : z_1] \in \mathbb{P}^2$. The set of all conics passing through this point form the hyperplane $H_1 \subset \mathbb{P}^5$ in variables a, b, c, d, e, f.

$$H_1 : \mathbb{V}({x_1}^2 a + {y_1}^2 b + {z_1}^2 c + x_1 y_1 d + x_1 z_1 e + y_1 z_1 f)$$

Similarly, the hyperplanes $H_2, H_3, H_4, H_5 \subset \mathbb{P}^5$ corresponds to the set of all conics that pass through respective points in $\mathbb{P}^2$. If five points in the plane are chosen such that no three of the points are collinear, then the hyperplanes $H_1, H_2, H_3, H_4, H_5 \subset \mathbb{P}^5$ are linearly independent by Lemma 10. This intersection of codimension-one linear subvarieties is a zero dimensional

linear space and thus contains one point in $\mathbb{P}^5$. Hence there is exactly one conic that passes through five points in $\mathbb{P}^2$. $\square$

4.4 The Degree of a Variety

A fundamental invariant of a projective variety is its degree. When dealing with plane algebraic curves the degree of a variety may be obvious from the magnitude of the exponents in its defining polynomial equations. We can think of the degree as the magnitude of the curvature. A curve of degree eight embedded in $\mathbb{P}^2$ will generally have more curvature, or bend, then a simple conic. It is important to develop a more rigorous notion of this property so that we can determine and discuss the degree of a variety generated by several complicated polynomial equations.

Definition 12. The degree of the projective variety V in $\mathbb{P}^n$ is the greatest possible finite number of intersection points of V with a linear subvariety $L \subset \mathbb{P}^n$ of dimension equal to the codimension of V.

This definition captures our intuition quite nicely. The maximal number of intersection points of some variety V and a linear subvariety of the proper codimension will increase as the variety bends more and more. The degree of a conic in the projective plane is *two* since a codimension-one linear subvariety (a line) intersects the conic in at most two points. We can see this algebraically since a conic is described by a degree two polynomial. A line in $\mathbb{P}^2$ is degree-one since any other line intersects it at one point.

The degree of a conic and a linear subvariety is easily determined. We are, however, interested in the degrees of more complicated algebraic varieties. Just as the set of all conics passing through a point form a hyperplane in $\mathbb{P}^5$, the set of all conics tangent to a fixed line in $\mathbb{P}^2$ form a codimension-one subvariety in $\mathbb{P}^5$. Ascertaining the degree of this subvariety is the first step to understanding conics tangent to lines.

Lemma 13. *Given a line $L : \mathbb{V}(Ax + By + Cz) \subset \mathbb{P}^2$, the set of all conics tangent to L form a degree-two hypersurface $T_L \subset \mathbb{P}^5$.*

Proof. Begin by considering the line L defined by the equation $z = 0$ and a general conic

$$C = \mathbb{V}(ax^2 + by^2 + cz^2 + dxy + exz + fyz).$$

The restriction of the conic C to line L is denoted $C\mid_L$ and is given by the equation

$$C\mid_L\colon ax^2 + by^2 + dxy = 0.$$

We de-homogenize the equation to a new variable $t = x/y$ and are left with

$$at^2 + dt + b = 0.$$

Solving this equation for t yields

$$t = \frac{-d \pm \sqrt{d^2 - 4ac}}{2a}.$$

The conic C is tangent to L when the discriminant of the above expression is zero. Thus the hypersurface formed by the condition that a conic is tangent to the line $z = 0$ is described by the degree 2 equation

$$d^2 - 4ac = 0.$$

We can choose a specific line L without a loss of generality since any two lines in $\mathbb{P}^2$ differ only by a linear change of coordinates, and this will not change our degree. □

The *Five Points Theorem* established that there is a unique conic that passes through five fixed points in $\mathbb{P}^2$. With the help of Lemma 13 we can tackle slightly more complicated enumerative problems.

Theorem 14. *Given any four non-collinear points $P_1, P_2, P_3, P_4 \in \mathbb{P}^2$ and any line $L_1 \subset \mathbb{P}^2$ there exist at most two non-degenerate conics that contain the four fixed points and are tangent to the fixed line.*

Proof. Consider first the set of all conics tangent to the fixed line L_1. This condition forms the degree-two hypersurface $S_1 \subset \mathbb{P}^5$. If we now consider the set of all conics that pass through the points P_1, P_2, P_3, P_4 we see that this constraint forms an intersection of hyperplanes

$H_1 \cap H_2 \cap H_3 \cap H_4 \subset \mathbb{P}^5$. This intersection, call it S_2, is a one dimensional surface of degree-one. The intersection $S_1 \cap S_2 \subset \mathbb{P}^5$ contains points corresponding to the set of conics that contain P_1, P_2, P_3, P_4 and are tangent to L_1. Since S_2 is a one dimensional line and S_1 is a codimension-one surface of degree-two, we know that the intersection $S_1 \cap S_2$ must contain at most two points. Therefore there are at most two conics that satisfy the set conditions. Later on we will see that in general, there are precisely two. □

At the end of the proof of theorem 14 we made use of the fact that a line intersects a degree-two surface at two points. This is quite clear since we are dealing with a linear variety. Bezout's Theorem is a famous result in algebraic geometry that allows us to further understand the intersection of two algebraic varieties when we have knowledge of their respective degrees. For a proof of this theorem see Shafarevich [S, p.173].

Before we proceed to the theorem we need to define an important term that will be used throughout the paper. The geometric locus of points on lines tangent to a projective variety X at the point x is called the tangent space to X at x. It is denoted by $\Theta_{X,x}$. Varieties $Y_1, \ldots, Y_r$ are said to intersect *transversely* at a point $x \in \bigcap Y_i$ if

$$codim_{\Theta_{X,x}}(\bigcap_{i=1}^{r} \Theta_{Y_i,x}) = \sum_{i=1}^{r} codim_X Y_i.$$

[S]

Theorem 15 ([S]). *If n hypersurfaces of degrees $d_1, d_2, \ldots, d_n$ intersect transversely in $\mathbb{P}^n$ then the intersection consists of $((d_1)(d_2)\cdots(d_n))$ points.*

This result provides a fundamental tool for determining the number of points of intersection of two algebraic varieties. When working in $\mathbb{P}^5$, the intersection of five codimension-one subvarieties will yield a finite number of points. In this case intersection consists of a number of points equal to the the product of the degrees of the varieties. Each point corresponds to a conic in the plane. However if the codimension of the intersection space is less than five, then Bezout's Theorem tells us the degree of the space of intersection, but it is not possible to interpret this number in terms of conics in the plane.

Theorem 16. *Given three points $P_1, P_2, P_3 \in \mathbb{P}^2$ and two lines $L_1, L_2 \subset \mathbb{P}^2$, there exist at most four conics that pass through the three points and are tangent to the two lines.*

Proof. The set of all conics that pass through the three fixed points in the plane corresponds to the intersection of hyperplanes $H_1 \cap H_2 \cap H_3 \subset \mathbb{P}^5$. We call this intersection S_1. By Lemma 10 it is clear that S_1 is a degree-one linear surface of dimension-two. The set of conics that are tangent to L_1 and the set of conics tangent to L_2 form hypersurfaces S_2 and S_3 respectively. The intersection $S_1 \cap S_2 \cap S_3 \subset \mathbb{P}^5$ is the set of all degree-two curves that satisfy our given constraints. $S_2 \cap S_3 \subset \mathbb{P}^5$ is a degree four variety of dimension-three. By intersecting S_1, S_2 and S_3 we form a zero dimensional space satisfying the given constraints. By invoking Bezout's Theorem we observe that $S_1 \cap S_2 \cap S_3$ contains four distinct points. Each point corresponds to a conic, so there are four conics satisfying the conditions. $\square$

The set of all conics that are tangent to a fixed conic form a hypersurface in $\mathbb{P}^5$. This surface however is very different than the one formed by conics tangent to a fixed line. The following lemma establishes the degree of another very important subvariety in $\mathbb{P}^5$.

Lemma 17. *Let Q be a conic in $\mathbb{P}^2$. The set of all conics tangent to Q form a degree six hypersurface in $\mathbb{P}^5$.*

Proof. Without loss of generality we can fix the conic $Q : xz - y^2 = 0$ in $\mathbb{P}^2$. We dehomogenize this conic to $x - y^2 = 0$. Similar to the proof of Lemma 13 we restrict the general dehomogenized conic $ax^2 + by^2 + cxy + dx + ey + f = 0$ to Q by substituting y^2 for x in the general equation. This yields the equation

$$ay^4 + by^2 + cy^3 + dy^2 + ey + f = 0.$$

A conic is tangent to Q when this polynomial and its derivative

$$4ay^3 + 2by + 3cy^2 + 2dy + e = 0$$

have a common root. Two polynomials have a common root when the *resultant* of the two is zero [CLO]. We computed the resultant using the computer algebra program *Maple.*

This computation produced a large degree-six polynomial in the variables a, b, c, d, e, f. This hypersurface vanishes over the set of all conics that are tangent to Q. □

4.5 Duality

At this point its important to introduce a fundamental correspondence between points and lines in the projective plane. If we consider both points and lines in the projective plane we recognize that the two geometric objects are *dual* elements of one another. Just as a line can be thought of as an uncountable stretch of points, we can view a point in the plane as an intersection of an infinite number of lines.

Consider the equation describing a line in $\mathbb{P}^2$:

$$ax + by + cz = 0,$$

where $a, b, c \in \mathbb{C}$ are constant coefficients. If a point $[x_0 : y_0 : z_0] \in \mathbb{P}^2$ lies on the line, then the following condition must be satisfied:

$$ax_0 + by_0 + cz_0 = 0.$$

A simple rearrangement shows that if we fix $[x_0 : y_0 : z_0]$ then this condition forms a line in the variables a, b, c:

$$x_0 a + y_0 b + z_0 c = 0.$$

We see that there is a one dimensional space of lines in $\mathbb{P}^2$ passing through the point $[x_0 : y_0 : z_0]$ [SC]. These observations lead us in a natural way to the following definition.

Definition 18. For any projective space $\mathbb{P}^n$, the *dual* of $\mathbb{P}^n$, denoted $\check{\mathbb{P}}^n$, is the moduli space of all hyperplanes in $\mathbb{P}^n$. A hyperplane in $\mathbb{P}^n$ with coefficients $y_0, y_1, ..., y_n$ is regarded as a point $[y_0 : y_1 : ... : y_n]$ in $\check{\mathbb{P}}^n$.

The idea of duality applies not only to statements involving points and linear subspaces, but also to non-linear algebraic varieties. The following result illustrates the important relationship between a conic in $\mathbb{P}^2$ and the dual projective plane.

Lemma 19. *Given a non-singular conic $Q \subset \mathbb{P}^2$, there exists a* dual *non-singular conic $\check{Q} \subset \check{\mathbb{P}}^2$ where the set of lines tangent to Q form the conic $\check{Q}$. Moreover, the set of all lines tangent to $\check{Q}$ correspond to the locus of points that form the conic Q.*

Proof. We begin with a conic $Q : ax^2 + by^2 + cz^2 + dxy + exz + fyz = 0$ and a line $L : Ax + By + Cz = 0$. Since one of A, B, C is not equal to zero, without loss of generality we can assume that $A \neq 0$. By rearranging terms in the equation of the line we can solve for the x variable and then restrict our general conic Q to L:

$$Q\mid_L: (\frac{aB^2}{A^2} - \frac{dB}{A} + b)y^2 + (\frac{2azCB}{A^2} - \frac{dC}{A} - \frac{eB}{A} + f)yz + (\frac{aC^2}{A^2} - \frac{eC}{A} + c)z^2.$$

The expression can now be dehomogenized to a new variable y/z yielding a quadratic equation. A conic and a line are tangent to each other when the *discriminant* of this quadratic equation vanishes, or when

$$(\frac{2azCB}{A^2} - \frac{dC}{A} - \frac{eB}{A} + f)^2 - 4(\frac{aB^2}{A^2} - \frac{dB}{A} + b)(\frac{aC^2}{A^2} - \frac{eC}{A} + c) = 0.$$

After expanding this expression and appropriately factoring the numerator we have the following equation:

$$(f^2-4bc)A^2+(4dc-2ef)BA+(4be-2df)CA+(e^2-4ac)B^2+(4af-2de)CB+(d^2-4ba)C^2 = 0.$$

This vanishing set is a conic $\check{Q}$ in the variables A, B, C and corresponds to the set of all lines tangent to Q. Figure 3 will help the reader to visualize this duality.

Now we show that $\check{\check{Q}} = Q$. Begin with our conic Q and its dual $\check{Q}$. Let P be a point on Q and let L denote the tangent line to Q at P. Then $\check{L}$ is a point on $\check{Q}$ and $\check{P}$ is a line through $\check{L}$. Suppose $\check{P}$ is not tangent to $\check{Q}$. Then $\check{P}$ intersects $\check{Q}$ at another point $\check{L}_2$ whose dual is a line L_2. L_2 goes through P, since $\check{L}_2$ is on $\check{P}$. Since $\check{L}_2$ is on $\check{Q}$, L_2 must be tangent to Q. Since L_2 goes through P and is tangent to Q,

$$L_2 = L \Rightarrow \check{L}_2 = \check{L}.$$

This implies that $\check{L}_2$ is tangent to $\check{Q}$. Now we can show that $\check{\check{Q}} = Q$.

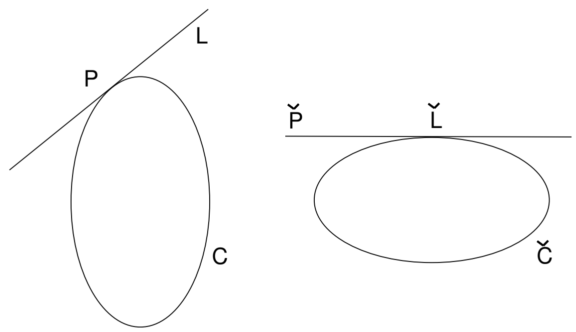

Fig. 3: Duality

Fix Q in the plane and mark five points that lie on the conic. We showed that each of these points gets taken to a unique line tangent to $\check{Q}$. The dual of these five lines is a set of five points in $\mathbb{P}^2$. By the five points theorem there is one conic that passes through them and hence $\check{\check{Q}} = Q$.

□

We can immediately make use of this lemma to prove a rather counterintuitive result. The five points theorem showed that we can determine exactly one conic by fixing five points in $\mathbb{P}^2$. The following theorem shows that a unique conic can also be determined by five lines in the projective plane.

Theorem 20. *Given five fixed lines in general position in $\mathbb{P}^2$ there is exactly one conic that is tangent to all five of them.*

Proof. By the principle of *duality* we know that the five fixed lines in $\mathbb{P}^2$ can be regarded as five points in $\check{\mathbb{P}}^2$. Theorem 1 tells us that there is one conic $\check{C}$ that passes through these points. Invoking Lemma 19 we see there is one conic C that is tangent to five fixed lines in $\mathbb{P}^2$. □

4.6 Excess Intersection

The proof of Theorem 20 is of a completely different form than the proofs for Theorems 14 and 16. The argument does not make use of the hypersurfaces formed under the condition

that a conic be tangent to a fixed line. In fact, if we were to attempt to prove Theorem 20 by considering the intersection of the five hypersurfaces formed by the fixed lines in $\mathbb{P}^2$, then we would be led to an erroneous conclusion.

The variety formed in $\mathbb{P}^5$ by the set of all conics tangent to a line is degree-two by Lemma 13. Naively applying Bezout's Theorem we assume that the intersection of the hypersurfaces contains $(2)(2)(2)(2)(2) = 2^5 = 32$ points in $\mathbb{P}^5$. This clearly contradicts the valid proof of Theorem 20, given above. We must check the hypothesis of the theorem carefully. This discrepancy is an elementary example of *Excess Intersection*.

The double line introduced in Example 8 is fundamental to our understanding of enumerative problems involving conics. When we ask questions involving tangency and plane conics, how do we take into account that every double line conic is tangent to every other curve in the plane? By considering each double line conic in $\mathbb{P}^2$ by its representative point in $\mathbb{P}^5$, we can view the set of all double line conics as a surface in our moduli space. By better understanding this variety we can hope to gain insight into how degenerate conics are affecting our counting. We will examine this variety in the following section.

5 Varieties Within the Moduli Space of Conics

This section is dedicated to describing the two most important subvarieties that lie within $\mathbb{P}^5$, the Veronese surface, and the Segre embedding. We examine how these geometric objects relate to problems of enumeration involving plane conics.

5.1 The Veronese Surface

The Veronese surface is the variety in $\mathbb{P}^5$ that corresponds to the set of all double line conics in the plane. Consider the double line conic $\mathbb{V}((Ax + By + Cz)^2)$.We see that

$$\mathbb{V}((Ax + By + Cz)^2) = \mathbb{V}(A^2x^2 + B^2y^2 + C^2z^2 + 2ABxy + 2ACxz + 2BCyz).$$

Thus the Veronese surface of double lines is the image of the mapping

$$\mathbb{P}^2 \longrightarrow \mathbb{P}^5,$$

$$[A : B : C] \overset{V}{\longmapsto} [A^2 : B^2 : C^2 : 2AB : 2AC : 2BC].$$

To show that this mapping is one-to-one we consider two points

$$[{A_1}^2 : {B_1}^2 : {C_1}^2 : 2A_1B_1 : 2A_1C_1 : 2B_1C_1], [{A_2}^2 : {B_2}^2 : {C_2}^2 : 2A_2B_2 : 2A_2C_2 : 2B_2C_2] \in \mathbb{P}^5$$

and assume that they are equal. Therefore the following equalities must hold

$${A_1}^2 = \lambda {A_2}^2, {B_1}^2 = \lambda {B_2}^2, {C_1}^2 = \lambda {C_2}^2$$

for some $\lambda \neq 0$

$$2A_1B_1 = \lambda 2A_2B_2, 2A_1C_1 = \lambda 2A_2C_2, 2B_1C_1 = \lambda 2B_2C_2.$$

From these conditions it is not difficult to show that

$$A_1 = \sqrt{\lambda}A_2, B_1 = \sqrt{\lambda}B_2, C_1 = \sqrt{\lambda}C_2.$$

Multiplication by a scalar does not change the point in projective space so our map is one-to-one. Our mapping is therefore an *embedding* of $\mathbb{P}^2$ into $\mathbb{P}^5$.

This map provides a parametrization of the surface in $\mathbb{P}^5$, but we can derive a set of six polynomial equations in variables a, b, c, d, e, f that vanish on this subvariety. We can see from the above parametrization that a point lying on the Veronese surface must satisfy the following equations

$$f^2 - 4bc = 0, e^2 - 4ac = 0$$

$$4dc - 2ef = 0, 4af - 2de = 0$$

$$4be - 2df = 0, d^2 - 4ba = 0.$$

These six polynomials generate what is known as a *radical ideal*, which is the algebraic representation of this subvariety in $\mathbb{P}^5$.

5.2 The Segre Variety

The image of the Segre embedding is the moduli space of projective crossed line conics that are embedded in $\mathbb{P}^2$. The variety is the embedding

$$\check{\mathbb{P}}^2 \times \check{\mathbb{P}}^2 \longrightarrow \mathbb{P}^5.$$

The Veronese variety is contained within the Segre. This conforms with our intuition since we can view the set of double line conics as a subset of the crossed lines. A double line is simply a pair of crossed lines that have come together and now lay on top of one another.

We can parameterize the Segre just as we parameterized the Veronese surface. Consider the pair of crossed lines

$$(aX + bY + cZ)(dX + eY + fZ)$$

that are embedded in $\mathbb{P}^2$. When we expand this polynomial out and drop the variables, we are left with the appropriate parametrization

$$[ad : be : cf : ae + bd : af + cd : bf + ce] \in \mathbb{P}^5.$$

The remainder of this section is dedicated to establishing the degree of the Segre. Lemma 21 and Lemma 22 will be a necessary lead up to the this important result. Since the Segre is a codimension-one subvariety inside of $\mathbb{P}^5$, it is very difficult to visualize. Knowing the degree of the variety will give a large amount of insight into how the set of points is *shaped* within our moduli space.

Lemma 21. *For every conic Q in $\mathbb{P}^2$ there is a set of conics that are tangent to Q. These conics form a dimension-four subvariety T_Q in $\mathbb{P}^5$. For any point $S \in T_Q$ the line that passes through the points S and Q is completely contained in T_Q. This is equivalent to saying that any point in $\mathbb{P}^5$ that lies on the line joining Q and S represents to a conic that is tangent to Q.*

Proof. Without loss of generality we can let R be the projective conic $xz - y^2 = 0$ and let T_R denote the variety in $\mathbb{P}^5$ that corresponds to the set of all conics that are tangent to R.

Choose a point $S : [a : b : c : d : e : f]$ that is contained in T_R. We describe the line L_{RS} containing both R and S with the following parametrization:

$$L_{RS} : [(1-t)a_1 : -t + (1-t)b_1 : (1-t)c_1 : (1-t)d_1 : t + (1-t)e_1 : (1-t)f_1].$$

Thus a conic lies on this line if it satisfies:

$$(1-t)(a_1)x^2+(-t+(1-t))(b_1)y^2+(1-t)(c_1)z^2+(1-t)(d_1)xy+(t+(1-t))(e_1)xz+(1-t)(f_1)yz = 0.$$

We need to check if this conic is tangent to R. We do so by dehomogenizing the above curve and restricting it to R to yield

$$(1-t)(a_1)y^4 + (1-t)(d_1)y^3 + (1-t)(b_1 + e_1)y^2 + (1-t)(f_1)y + (1-t) = 0.$$

We then differentiate the polynomial to obtain

$$(1-t)(4a_1)y^3 + (1-t)(3d_1)y^2 + (1-t)(2b_1 + 2e_1)y + (1-t)f_1 = 0.$$

These two polynomials have a common root when the determinant of the following Sylvester Matrix vanishes

$$(1-t)\begin{pmatrix} a & 0 & 0 & 4a & 0 & 0 & 0 \\ c & a & 0 & 3c & 4a & 0 & 0 \\ d+b & c & a & 2b+2d & 3c & 4a & 0 \\ e & d+b & c & e & 2b+2d & 3c & 4a \\ f & e & d+b & 0 & e & 2b+2d & 3c \\ 0 & f & e & 0 & 0 & e & 2b+2d \\ 0 & 0 & f & 0 & 0 & 0 & e \end{pmatrix}.$$

But since the conic S was chosen to be tangent to R we know that the determinant of this matrix vanishes by Lemma 17 and the definition of the *resultant*. Thus every point on the line $L_R S$ corresponds to a conic that is tangent to R.

□

Lemma 22. *A generic line in $\mathbb{P}^5$ is the intersection of four linearly independent hyperplanes, each corresponding to the set of plane conics that pass through a fixed point in $\mathbb{P}^2$.*

Proof. Let L be a line in $\mathbb{P}^5$ and let $[a_0 : b_0 : c_0 : d_0 : e_0 : f_0]$ and $[a_1 : b_1 : c_1 : d_1 : e_1 : f_1]$ be two points that lie on this line. Consider their corresponding conics in $\mathbb{P}^2$. In general the two conics will intersect in four distinct points P_1, P_2, P_3 and P_4. Now consider the set of all conics that pass through one of these intersection points, say P_1. Each of the two conics are members of this set. Since the set of all conics passing through P_1 forms a hyperplane $H_1 \in \mathbb{P}^5$, our two points $[a_0 : b_0 : c_0 : d_0 : e_0 : f_0]$ and $[a_1 : b_1 : c_1 : d_1 : e_1 : f_1]$ are contained in H_1 as well as each of the hyperplanes H_2, H_3 and H_4. We know that four linearly independent hyperplanes intersect in a line and we know that this is precisely the line L.

We must show that we can choose a generic line in $\mathbb{P}^5$ such that pairs of points on the line correspond to conics that intersect in four distinct points. We compute the dimension of the set of lines in $\mathbb{P}^5$ as follows. We have 5 dimensions of freedom to choose a point in $\mathbb{P}^5$. Two points determine a line in $\mathbb{P}^5$. However, each of the two points has a one dimensional space of freedom to move and yet still determine the same line. This freedom of movement must be subtracted from the total dimensionality. Thus the dimension of the space of all lines in $\mathbb{P}^5$ is $5 + 5 - 2 = 8$.

By Lemma 21 we know that we can choose a line in $\mathbb{P}^5$ that is completely contained in a subvariety that is formed by the set of all conics that are tangent to some conic that is contained in our chosen line. Such a line would not be generic and would be insufficient for determining the degree of a codimension-one subvariety in $\mathbb{P}^5$. We need to show that it is possible to choose a line in $\mathbb{P}^5$ that is not contained in the above subvariety. We have five dimensions of freedom to choose a point Q in $\mathbb{P}^5$. Upon choosing this point we have four dimensions of freedom in choosing a point that determines a line which is completely contained in the variety T_Q. When choosing two points to determine a line, there is a two dimensional space of points that fix the same line. Thus the dimension of the space of lines, each of which corresponds to a set of conics that are tangent to one another, is $5+4-2 = 7$.

Since the space of lines in $\mathbb{P}^5$ is eight dimensional we have the freedom to choose a generic line that does not correspond to set of conics in the plane that are tangent to one another.

□

Lemma 23. *The degree of the Segre Variety in $\mathbb{P}^5$ is three.*

Proof. From the definition of degree we know that the degree of the Segre variety in $\mathbb{P}^5$ is equivalent to the number of points on the Segre lying on a generic line in $\mathbb{P}^5$. Choose a generic line in $\mathbb{P}^5$. By lemma 22 we know that we can choose four points in the plane such that conics passing through these points correspond to linearly independent hyperplanes that intersect on our chosen line. This line will intersect the Segre variety precisely when a singular conic in $\mathbb{P}^2$ contains the four fixed points. There are only three possible pairs of crossed lines that can contain four points in the plane. Thus the degree of the Segre variety is three.

□

6 The Blowup of $\mathbb{P}^5$

6.1 Definition of the Blowup

From the equations of the six hypersurfaces that intersect on the Veronese we are able to define the following birational morphism F on $\mathbb{P}^5$:

$$\mathbb{P}^5 \xrightarrow{F} \mathbb{P}^5$$

$$[a : b : c : d : e : f] \to [f^2 - 4bc : e^2 - 4ac : d^2 - 4ba : 4dc - 2ef : 4be - 2df : 4af - 2de]$$

We see that F is not well defined for the set of points in $\mathbb{P}^5$ that lie on the Veronese surface. Points in $\mathbb{P}^5$ that correspond to double line conics are mapped to the vector $[0 : 0 : 0 : 0 : 0 : 0]$, which is not a point in projective space.

Definition 24. If we let F be the above function, we define the *Blowup* of $\mathbb{P}^5$ along the Veronese surface, denoted $Bl_V(\mathbb{P}^5)$, to be the closure of the graph of the map F in $\mathbb{P}^5 \times \mathbb{P}^5$.

$Bl_V(\mathbb{P}^5)$ can be thought of as a stretching of the moduli space conics. $Bl_V(\mathbb{P}^5)$ preserves all properties of non-singular and crossed line conics, but adds additional information to the Veronese surface.

Definition 25. The *Exceptional divisor* E, in $Bl_V(\mathbb{P}^5)$ is the set of points added by taking the closure of the graph of F. It is an irreducible and codimension-one subvariety of $Bl_V(\mathbb{P}^5)$.

If we let V denote the Veronese surface in $\mathbb{P}^5$, then the projection map

$$Bl_V(\mathbb{P}^5) \xrightarrow{\pi} \mathbb{P}^5$$

$$(x, F(x)) \mapsto x$$

defines an isomorphism between the open sets

$$Bl_V(\mathbb{P}^5) \setminus \pi^{-1}(V) \to \mathbb{P}^5 \setminus V.$$

If Z is an irreducible variety in $\mathbb{P}^5$ then $\pi^{-1}(Z)$ is what is referred to as the *total transform* of Z. If $\pi^{-1}(Z)$ is reducible and has a multiple of the exceptional divisor, mE as a component, then $\pi^{-1}(Z) \setminus mE$ is called the *proper transform* of Z [H]. The closure of the set of points on the variety $\pi^{-1}(Z)$ that lie off of the exceptional divisor is referred to as the strict transform of Z.

The isomorphism described above fits in nicely with our intuition about the blowup of $\mathbb{P}^5$. Conics lying outside of the Veronese are unchanged by blowing up the space.

6.2 The Exceptional Divisor

The exceptional divisor E is a codimension-one variety in $Bl_V(\mathbb{P}^5)$ and is the preimage $\pi^{-1}(V)$ of the Veronese surface. We defined the blowup of $\mathbb{P}^5$ along the Veronese to be the closure of the graph of a map defined by the generating polynomials of the Veronese embedding. The exceptional divisor can be thought of as the set of points on the graph that are added to close out the space. Every point on the exceptional divisor is therefore the limit of a sequence of points that lie off of this variety. So by examining a sequence of points that

converge to a point in the closure, we can better understand the set of points that form the exceptional divisor.

There is a large open set NS in $Bl_V(\mathbb{P}^5)$ that lies outside of the exceptional divisor. The coordinates of each point $p \in NS$ are given by a row vector with twelve entries. The first six entries are coefficients of a smooth conic $C \subset \mathbb{P}^5$ while the second six entries are the coefficients of the smooth conic $\check{C} \subset \check{\mathbb{P}}^5$.We know this because the polynomials that generate the ideal which corresponds to the Veronese surface are precisely the polynomials that define the coefficients of the dual conic.

E is the preimage of the Veronese under our map

$$Bl_V(\mathbb{P}^5) \xrightarrow{\pi_1} \mathbb{P}^5$$

$$(c_1, c_2) \longmapsto (c_1).$$

So we know precisely the first factor of the coordinate describing a point $p \in E$. The first six entries are the coefficients of a double line conic in $\mathbb{P}^2$. The second factor of the point is not so easy to discern since within $\mathbb{P}^5$, duality is not well defined for points on the Veronese. We can attack this problem by making use of the definition of *set closure*.

Consider a sequence of points $\{p_i\} \in NS$ that converges to a point $p \in E$. The first factor of each element in the sequence is composed of the coefficients of some non-singular conic C_i, while entries in the second factor are the coefficients of the non-singular conic $\check{C}_i$. By the definition of a graph we know that the first factor converges to the coefficients of a point on the Veronese. This can be thought of as the deformation of a smooth conic in $\mathbb{P}^2$ into a double line. At each instant of this deformation we have a conic in the plane that has a well defined dual. The limit of this sequence of dual conics $\{\check{C}_i\}$ as the sequence $\{C_i\}$ approaches a double line, is a conic whose coefficients are the last six entries of the coordinate describing the point p.

Now we will evaluate this limit. We begin by choosing a set of non-singular conics that can be deformed into a double line. Let V be the image of the Veronese embedding in $\mathbb{P}^5$ and choose a point $v \in V$. v is a point of the form $[A^2 : B^2 : C^2 : 2AB : 2AC : 2BC]$

where $A, B, C \in \mathbb{C}$. Now consider the line segment $L_{vq} \subset \mathbb{P}^5$ that connects v to some point $q = [a : b : c : d : e : f]$ which corresponds to a non-singular conic $Q \in \mathbb{P}^2$. We can maneuver along this line segment with the following parametrization:

$$q(t) : [A^2(1-t)+at : B^2(1-t)+bt : C^2(1-t)+ct : 2AB(1-t)+dt : 2AC(1-t)+et : 2BC(1-t)+ft]$$

where the parameter t is a real number ranging from zero to one. As t approaches zero, the family of non-singular conics degenerates into a double line, and $q(0) = v$. We have already established by lemma 19 the following mapping which takes a point in $\mathbb{P}^5$ to its dual point

$$\mathbb{P}^5 \longrightarrow \check{\mathbb{P}}^5$$

$$[a : b : c : d : e : f] \longmapsto [(f^2-4bc) : (e^2-4ac) : (d^2-4ba) : (4dc-2ef) : (4be-2df) : (4af-2de)].$$

The limit of $\check{q}(t)$ as t approaches zero is the second factor in the coordinate describing a point on E. The computation of this limit can be done using the computer algebra system *Maple*. We can now regard this limit point in $\mathbb{P}^5$ as a conic in the variables X, Y, Z. This conic factors giving

$$\mathbb{V}([\sqrt{c}(B-\frac{f}{2c}C-\sqrt{\frac{f^2-4bc}{4c^2}}C)X+\sqrt{a}(C-\frac{e}{2a}A-\sqrt{\frac{e^2-4ac}{4a^2}}A)Y+\sqrt{b}(A-\frac{d}{2b}B-\sqrt{\frac{d^2-4ab}{4b^2}}B)Z]$$
$$[\sqrt{c}(B-\frac{f}{2c}C+\sqrt{\frac{f^2-4bc}{4c^2}}C)X+\sqrt{a}(C-\frac{e}{2a}A+\sqrt{\frac{e^2-4ac}{4a^2}}A)Y+\sqrt{b}(A-\frac{d}{2b}B+\sqrt{\frac{d^2-4ab}{4b^2}}B)Z]).$$

We see that the second factor of a point on E is the coefficients of a crossed line conic, or a point on the Segre embedding.

6.3 Duality and The Blowup

We can now construct a bijection from the exceptional divisor to $\mathbb{P}^5$ with the following mapping:

$$Bl_V(\mathbb{P}^5) \longrightarrow \mathbb{P}^5$$

$$[a : b] \in E \longmapsto b \in \mathbb{P}^5$$

where a is a point on the Veronese and b is a point on the Segre such that $\check{b}$ equals a.

This bijection shows us that the second factor of E is isomorphic to the variety of crossed line conics in $\mathbb{P}^5$. Therefore, a point on the exceptional divisor can be described by choosing a point on the Segre variety and then computing the dual of that point.

The dual map introduced in Lemma 19 is well defined for all points in $\mathbb{P}^5$ that lie off of the Veronese surface, because every non-singular and crossed line conic is taken to a unique dual. The map is not well defined for points on the Veronese since a point corresponding to a double line conic is mapped to $[0:0:0:0:0:0]$. We have just seen that blowing the space of conics seems to complete the dual mapping. A point v on the Veronese surface is now identified with the set of all crossed lines that are sent to v under the mapping introduced in Lemma 19.

7 Resolution of Steiner's Problem

7.1 The Chow Ring and Rational Equivalence

Before we attempt to resolve further enumerative problems we need to develop a formal set of algebraic operations that will allow us to work with the underlying geometry of the blowup of $\mathbb{P}^5$. We begin the construction of these algebraic operations by first examining an important property of subvarieties in projective space. The property of rational equivalence gives algebraic geometers a method of classifying two subvarieties that may look different, but share many essential intersection properties.

Definition 26. A codimension-k cycle on X is a finite formal sum of irreducible codimension-k subvarieties of a projective space X. We denote by $Z^k(X)$ the set of all codimension-k cycles of X.

Both the zeros and the poles of a rational function are cycles. For example, a zero of order three corresponds to three copies of a linear subvariety.

We see that each element of Z^kX is a finite linear combination irreducible subvarieties

$$a_1Y_1 + a_2Y_2 + \cdots a_nY_n$$

such that each a_i is an integer and each Y_i is an irreducible subvariety of X of codimension k.

We introduce the operation of addition to Z^kX by

$$\sum_{i=1}^{m} a_iY_i + \sum_{i=1}^{n} b_iY_i = \sum_{i} (a_i + b_i)Y_i.$$

The zero element can be thought of as the empty set. For each $\sum_{i=1}^{m} a_iY_i \in Z^kX$ there is an element $-\sum_{i=1}^{m} a_iY_i$ such that $\sum_{i=1}^{m} a_iY_i + (-\sum_{i=1}^{m} a_iY_i) = 0$. Associativity holds within the set since the addition of integers is associative. Thus $Z^k(X)$ is a group under the operation of addition.

Definition 27. Let Y_1 and Y_2 be codimension i cycles within an ambient space X. These two cycles are said to be *rationally equivalent* if there is a codimension $i-1$ cycle W of X containing Y_1 and Y_2, and a rational function f on W such that

$$Y_1 = \text{zeros of } f$$

and

$$Y_2 = \text{poles of } f.$$

We illustrate this concept with an example. Let $Y_1 = \mathbb{V}(y^2 + x^2)$ and $Y_2 = \mathbb{V}(yz - x^2)$ be conics in $\mathbb{P}^2$. Y_1 and Y_2 are rationally equivalent since $\mathbb{P}^2$ contains the curves and we can construct a rational function $f(x, y, z) = \frac{y^2+x^2}{yz-x^2}$ which clearly has Y_1 as its zeros and Y_2 as its poles. In fact, any two varieties of the same degree in $\mathbb{P}^n$ are rationally equivalent. This is an important point since we can now say that nonlinear varieties are rationally equivalent to linear varieties. For example, a hypersurface of degree three is rationally equivalent to three copies of a codimension-one linear cycle.

If Y is a cycle of some projective space, then the set of all cycles that are rationally equivalent to Y is called the *rational equivalence class* of Y, and is denoted $[Y]$. We are now ready to construct what is known as the *Chow Ring*.

It is easily show that the cycles that are rationally equivalent to zero form a subgroup of $Z^k(X)$.

Definition 28. If X is a projective space of dimension n we define A^kX to be the quotient of $Z^k(X)$ by the subgroup of cycles that are rationally equivalent to zero. Thus $A^k(X)$ will be the group of linear combinations of rational equivalence classes of codimension k cycles. We define $A^*X = \bigoplus_{k=0}^{5} A^kX$.

Multiplication of elements in A^*X reflects the way that cycles intersect in $\mathbb{P}^n$. We can now introduce the formal multiplication operation of A^*X.

Definition 29. We define the multiplication of two elements $[Y_1] \in A^kX$ and $[Y_2] \in A^lX$ as $[Y_1] \bullet [Y_2] \in A^{k+l}X = [Y_\alpha \cap Y_\beta]$ such that there are two subvarieties $Y_\alpha \in [Y_1]$ and $Y_\beta \in [Y_2]$ that intersect *transversely.*

We see that the multiplicative identity in our ring is the rational equivalence class of the entire space X. This multiplication operation can be carried out if we can find two appropriate subvarieties that intersect transversely. But as we know, projective varieties do not always intersect transversely. We can only multiply two rational equivalence classes when we can choose two elements that are not tangent, or do not contain a common component. Fortunately, we know that we can always choose the appropriate cycles.

Lemma 30 ([F]). *Let Y_1 and Y_2 be subvarieties in a projective space X. There exist subvarieties Y_1' and Y_2' that are rationally equivalent to Y_1 and Y_2 respectively and intersect transversely.*

For a proof of this result we refer the reader to [F].

7.2 The Cohomology of the Blowup

We can establish a variety of results about rational equivalence classes of cycles in a large projective space that will allow us to understand subvarieties in both $\mathbb{P}^5$ and the blowup of

$\mathbb{P}^5$ along the image of the Veronese embedding. Let Z be an irreducible closed subset of a projective variety X and let $U = X \setminus Z$ be its complement.

Theorem 31 ([F]). *There exists a surjective homomorphism from A^1X to A^1U.*

Proof. Let D be a codimension-one cycle of a projective space X with $[D] \in A^1X$. We construct a map

$$g : A^1X \longrightarrow A^1U$$

$$[D] \xmapsto{g} [D \cap U].$$

Thus if $D = mZ$ where $m \in \mathbb{Z}$ then

$$D \xrightarrow{g} \emptyset.$$

We can show that the map is well defined. Let D_1 and D_2 be rationally equivalent cycles in X. We see that $g(D_1)$ and $g(D_2)$ are rationally equivalent since we can restrict a rational function f which has D_1 as its zeros and D_2 as its poles to the space $X \setminus Z$.

Our map g is a homomorphism since $\overline{[D_1 + D_2]} = [\bar{D_1}] + [\bar{D_2}]$ and $\overline{D_1 \bullet D_2} = \bar{D_1} \bullet \bar{D_2}$. To show that the map is surjective we consider a class $[F]$ of a representative cycle F of U. Let $\bar{F}$ be the closure of F in X. We see that

$$g[\bar{F}] = [F].$$

Hence the map g is surjective. □

Corollary 32. *Let Z be a codimension-one variety and assume that Z is irreducible in the ambient space X. The kernel of the map g above, which takes elements of A^1X to elements of $A^1(X \setminus Z)$, is $\mathbb{Z}[Z]$.*

Proof. It is clear that $\mathbb{Z}[Z]$ is in $ker(g)$. We need to show that $ker(g) \subset \mathbb{Z}[Z]$. Let $[W] \in A^1X$ such that $[W]$ is not an integer multiple of $[Z]$. If D is a cycle in $[W]$, then it is codimension-one and cannot be contained in Z and thus must contain points in $X \setminus Z$. We have therefore guaranteed that the map g takes points in $[W]$ to points on U and hence $[W] \notin ker(g)$. Thus $ker(g) \subset \mathbb{Z}[Z]$ and $ker(g) = \mathbb{Z}[Z]$. □

Corollary 33. *Let the codimension of Z be greater than one. The map introduced in theorem* 31 *is an isomorphism.*

Proof. Given a projective space X, the surjective map g takes elements from A^1X to elements in $A^1(X \setminus Z)$.

Let D_1, D_2 be codimension one cycles in $X \setminus Z$ such that D_1 and D_2 are rationally equivalent to each other, and let $\bar{D}_1$ and $\bar{D}_2$ be their respective closures in X.

Since g maps only elements of A^1X to $X \setminus Z$ any cycle D that we can pick is too big to be contained in Z. Thus the kernel of g is zero and our map g is injective and hence an isomorphism.

□

We now need to incorporate the above results into the context of $Bl_V(\mathbb{P}^5)$, the blowup of $\mathbb{P}^5$ along the image of the Veronese embedding. We let E be the closed exceptional divisor of $Bl_V(\mathbb{P}^5)$ and let U be its open complement.

Theorem 34. *$A^1(Bl_V(\mathbb{P}^5))$ is isomorphic to $A^1(U) \oplus \mathbb{Z}[E]$.*

Proof. By Theorem 31 we know that there exists a surjective map g that takes elements in $A^1(Bl_V(\mathbb{P}^5))$ to elements in $A^1(Bl_V(\mathbb{P}^5) \setminus E)$. Let f be the map that takes integer multiples of the exceptional divisor into $A^1(Bl_V(\mathbb{P}^5))$. Since E is codimension-one in $Bl_V(\mathbb{P}^5)$ we know that $Ker(g) = \mathbb{Z}[E]$ by corollary 32. (The only elements of $A^1(Bl_V(\mathbb{P}^5))$ that are not mapped to $A^1(Bl_V(\mathbb{P}^5) \setminus E)$ are multiples of $[E]$ itself).

Consider the following sequence of homomorphisms

$$0 \longrightarrow \mathbb{Z}[E] \xrightarrow{f} A^1(Bl_V(\mathbb{P}^5)) \xrightarrow{g} A^1(Bl_V(\mathbb{P}^5) \setminus E) \longrightarrow 0.$$

The kernel of the map f is zero since every integer multiple of the exceptional divisor is mapped into $A^1(X)$. We have established by corollary 32 that the kernel of the map g is the set of integer multiples of the exceptional divisor. Thus the image of f is equal to $ker(g)$. By Theorem 33, g is surjective. Our set of maps above is therefore a *short exact sequence.*

Because there is a homomorphism from $A^1(Bl_V(\mathbb{P}^5) \setminus E)$ to $A^1(Bl_V(\mathbb{P}^5))$, namely the morphism obtained by taking the closure of a divisor, we know that our short exact sequence *splits* and $A^1(Bl_V(\mathbb{P}^5)) = A^1(Bl_V(\mathbb{P}^5) \setminus E) \oplus \mathbb{Z}[E]$. □

Theorem 35 ([FS]). *Let H be a hyperplane in $\mathbb{P}^5$, and let $\tilde{H} = \pi^{-1}(H)$. The ring $A^1(Bl_V(\mathbb{P}^5))$ is isomorphic to $\mathbb{Z}[\tilde{H}] + \mathbb{Z}[E]$.*

Proof. By theorem 34 we know that

$$A^1(Bl_V(\mathbb{P}^5)) \cong A^1(Bl_V(\mathbb{P}^5) \setminus E) \oplus \mathbb{Z}E$$

and we have established in section 6.1 the isomorphism $A^1(Bl_V(\mathbb{P}^5) \setminus E) \cong A^1(\mathbb{P}^5 \setminus V)$. By invoking corollary 33 we see that $A^1(\mathbb{P}^5 \setminus V)$ is isomorphic to $A^1(\mathbb{P}^5)$. Each element of $A^1(\mathbb{P}^5)$ is an integer multiple of some general hyperplane class $[H]$. By linking this chain of isomorphisms we conclude

$$A^1(Bl_V(\mathbb{P}^5)) \cong \mathbb{Z}[\tilde{H}] \oplus \mathbb{Z}[E].$$

□

If two varieties X_1 and X_2 are rationally equivalent in $\mathbb{P}^5$, then their inverse images under our map π are rationally equivalent. We know this because there is a rational function, say g, with zeros X_1 and poles X_2. The function $g \circ \pi$ has $\pi^{-1}(X_1)$ as its zeros and $\pi^{-1}(X_2)$ as its poles. It follows that as cycles, $n\pi^{-1}(X_1) = \pi^{-1}(nX_1)$ where n is an integer.

Let T_Q be the set of all conics tangent to some fixed conic Q. Let $\tilde{H}$ equal $\pi^{-1}(H)$, where H is some arbitrary hyperplane in $\mathbb{P}^5$. $\pi^{-1}(T_Q) \sim \pi^{-1}(6H)$ and $\pi^{-1}(6H) \sim 6\tilde{H}$ So we now know that $\pi^{-1}(T_Q)$ is equivalent to $6\tilde{H}$.

We define $\tilde{T}_Q$ to be the strict transform of T_Q under the map π. $\pi^{-1}(T_Q)$ is not irreducible as a cycle since it can be broken into two components, namely $\tilde{T}_Q$ and nE where n is some integer. We will now prove a very important result.

Lemma 36. $\pi^{-1}(T_Q) = \tilde{T}_Q + 2E$

Proof. Let f_Q be the polynomial whose vanishing set is the variety T_Q. We first need to show that $\mathcal{I}(\pi^{-1}(T_Q)) \subset \mathcal{I}(E)^2$.

It can be shown using Macauley or a similar computer algebra system, that $\mathcal{I}(T_Q) \subset \mathcal{I}(V)^2$ and $\mathcal{I}(T_Q) \not\subset \mathcal{I}(V)^3$. Since f_Q generates $\mathcal{I}(T_Q)$, $f_Q \notin \mathcal{I}(V)^3$.

Now consider the zeros of the function $(f_Q \circ \pi)$ on the blowup $Bl_V(\mathbb{P}^5)$. For any point $\tilde{p} \in \pi^{-1}(T_Q)$,

$$\pi(\tilde{p}) \in T_Q \Longleftrightarrow f_Q(\pi(\tilde{p})) = 0 \Longleftrightarrow (f_Q \circ \pi)(\tilde{p}) = 0.$$

Therefore $(f_Q \circ \pi)$ generates $\mathcal{I}(\pi^{-1}(T_Q))$.
(1)

Because f_Q is in $\mathcal{I}(V)^2$ we can choose functions f_1, f_2 such that $f_1, f_2 \in \mathcal{I}(V)$ and $f_Q = f_1, f_2$. By the same reasoning as in 1, $f_1 \circ \pi$ and $f_2 \circ \pi$ are in $\mathcal{I}(\pi^{-1}(V)) = \mathcal{I}(E)$. Note that for any point $\tilde{p}$ in $Bl_V(\mathbb{P}^5)$,

$$(f_Q \circ \pi)(p) = f_Q(\pi(\tilde{p})) = f_1(\pi(\tilde{p}))f_2(\pi(\tilde{p})) = (f_1 \circ \pi)(\tilde{p})(f_2 \circ \pi)(\tilde{p}).$$

And thus, $(f_Q \circ \pi) = (f_1 \circ \pi)(f_2 \circ \pi)$.

We know then that $(f_Q \circ \pi) \subset \mathcal{I}(E)^2$. This implies that the zeros of $(f_Q \circ \pi)$ include $2E$. Note that neither f_1 nor f_2 can be in $\mathcal{I}(V)^2$, so the zeros of $(f_Q \circ \pi)$ do not include $3E$. Away from E, the zeros of $f_Q \circ \pi = \tilde{T}_Q$ by the definition of $\tilde{T}_Q$.

Thus, the zeros of $f_Q \circ \pi$ are $\tilde{T}_Q + 2E$. Hence $\pi^{-1}(T_Q) = \tilde{T}_Q + 2E$.

$\square$

$[\tilde{T}_Q] = 6[\tilde{H}] - 2[E]$

Proof. This follows from the fact that $\pi^{-1}(T_Q)$ is rationally equivalent to $6\tilde{H}$ and by Lemma 36 is rationally equivalent to $\tilde{T}_Q + 2E$.

□

By a similar proofs, we can show that the strict transform of the variety T_L, which is the set of all conics tangent to some line L, is rationally equivalent to $2\tilde{H} - E$, and also that $[\tilde{T_L}] = [2\tilde{H}] - [E]$.

7.3 Solution to Steiner's Problem

We now have all of the tools necessary to understand a key relationship between varieties in $Bl_V(\mathbb{P}^5)$. The following identity gives the fundamental relationship between cycles $\tilde{T_P}$,$\tilde{T_L}$ and $\tilde{T_Q}$.

Theorem 37. *The cycles $\tilde{T_P}, \tilde{T_L}$ and $\tilde{T_Q}$ on $Bl_V(\mathbb{P}^5)$ are related by the following identity:*

$$\tilde{T_Q} = 2\tilde{T_P} + 2\tilde{T_L}$$

Proof. We know that $\tilde{T_Q} = 6\tilde{T_P} - 2E$. It is also clear that $\tilde{T_L} = 2\tilde{T_P} - E$. By solving for E in this equation we obtain the following relation:

$$\tilde{T_Q} = 6\tilde{T_P} - 2(2\tilde{T_P} - \tilde{T_L}) = 2\tilde{T_P} + 2\tilde{T_L}.$$

□

We can make use of this powerful result and our previous theorems to solve a variety of enumerative problems involving conics and lines. We will now make frequent use of the multiplication operation defined for the Chow ring. Consider the multiplication of two divisors $(\tilde{T_L})^3$ and $(\tilde{T_P})^2$. Evaluating the expression

$$(\tilde{T_L})^3(\tilde{T_P})^2$$

is equivalent to asking the question: How many conics pass through two points and are tangent to three given lines? We know by the principle of duality and Theorem 16 that the answer is four.

Before we proceed with the final calculations to the various enumerative problems, we need to answer two central questions. Given the conics that satisfy the conditions to a particular variation of Steiner's problem: When are all of these conics non-singular? And in the case that they are all non-singular, is our solution set of conics composed of unique curves, or are we dealing with conics that have some multiplicity?

We first need to show that in general, there will be only *non-singular* conics tangent to five fixed non-singular conics. Let T_{Q_i} be the hypersurface in $\mathbb{P}^5$ formed by the set of all conics tangent to some non-singular conic $Q_i \subset \mathbb{P}^2$.

Lemma 38. *The set* $\{(Q_1, Q_2, Q_3, Q_4, Q_5)\} \subset (\mathbb{P}^5)^5$ *such that* $\bigcap_{i=1}^5 \tilde{T}_{Q_i}$ *consists only of isolated points, is open.*

Proof. We begin by defining a set

$$V = \{(Q_1, Q_2, Q_3, Q_4, Q_5, P) : P \in \bigcap_{i=1}^{5} \tilde{T}_{Q_i}\}$$

where $V \subset (\mathbb{P}^5)^5 \times Bl_V(\mathbb{P}^5)$. Also we define the function K such that $K(Q_1, Q_2, Q_3, Q_4, Q_5, P)$ is equal to the dimension of the tangent space to $\bigcap_{i=1}^5 \tilde{T}_{Q_i}$ at P. Let $\nabla F_1(P), \nabla F_2(P), \ldots, \nabla F_5(P)$ be the respective normal vectors to each of the five hypersurfaces at the point P. We know that any vector in the tangent space at P is perpendicular to this set of vectors. Any such vector is in the kernel of J, the Jacobian matrix defined as

$$\begin{pmatrix} \nabla F_1(P) \\ \nabla F_2(P) \\ \nabla F_3(P) \\ \nabla F_4(P) \\ \nabla F_5(P) \end{pmatrix}.$$

We know that $rank(J) + nullity(J)$ is equal to the number of variables. We can rearrange this equality to see that $K(P) = 6 - rank(J)$. We see here that the maximal rank of the matrix J is five. So therefore, if the matrix is of full rank then the function K will tell us that the dimension of the tangent space is one. It is one dimensional in affine space, but

when projectivized, the dimensionality is dropped to zero. We are interested in determining when the output of the function K is greater than or equal to one. If the output of the function K is greater than zero then we know that the multiplicity at the point is greater than one.

A well known result from linear algebra tells us that all of the maximal minors of J will vanish if and only if J is not of full rank. Each row within the matrix J is the gradient vector $\nabla F_i(P)$ to some hypersurface $\tilde{T}_{Q_i} \subset Bl_V(\mathbb{P}^5)$ at the point P. Given a conic Q_i in $\mathbb{P}^2$ the hypersurface $T_{Q_i} \subset \mathbb{P}^5$ formed by the set of all conics tangent to Q_i is the vanishing set of a polynomial in the standard variables a, b, c, d, e, f of $\mathbb{P}^5$. The hypersurface $\tilde{T}_{Q_i}$ is the strict transform of T_{Q_i} under our well defined projection map π.

Depending on the location of the point P in $Bl_V(\mathbb{P}^5)$, a particular gradient vector, $\nabla F_i(P)$ will be defined by a polynomial in the variables giving the coordinates of P with coefficients a, b, c, d, e, f. In short, each entry of our matrix J is a given by a polynomial. Thus the condition that the determinants of the 5×5 minors of our Jacobian matrix J vanish, is given by polynomial equations. It is a closed set in the Zariski topology.

This closed set lies within the space $(\mathbb{P}^5)^5 \times Bl_V(\mathbb{P}^5)$. This is not sufficient to complete the proof, since we must show that the image of our set V, under the projection to $(\mathbb{P}^5)^5$, is closed. Fortunately, we are working within projective space, where it is known that the projection from one space onto another maps closed varieties to closed varieties. Thus our closed variety within $(\mathbb{P}^5)^5 \times Bl_V(\mathbb{P}^5)$ remains closed when projected onto $(\mathbb{P}^5)^5$.

We have shown that the set $\{(Q_1, Q_2, Q_3, Q_4, Q_5)\} \subset (\mathbb{P}^5)^5$ such that there is a non-isolated point P in $\bigcap_{i=1}^5 \tilde{T}_{Q_i}$, is closed in $(\mathbb{P}^5)^5$. Thus the complement of this set is open.

We now need to show that this set is non-empty, since the empty set is defined to be open in the Zariski Topology. □

We will now show that the solution set for Steiner's Problem will in general consist only of non-singular conics.

Lemma 39. *The set $\{(Q_1, Q_2, Q_3, Q_4, Q_5)\} \subset (\mathbb{P}^5)^5$ such that $\bigcap_{i=1}^5 \tilde{T}_{Q_i}$ consists only of points lying off of the exceptional divisor, is open.*

Proof. Let V denote the set $\{(Q_1, Q_2, Q_3, Q_4, Q_5, P)\} : P \in \bigcap_{i=1}^{5} \tilde{T}_{Q_i}$. This set is closed in the space $(\mathbb{P}^5)^5 \times Bl_V(\mathbb{P}^5)$. Consider now the space $W : [(\mathbb{P}^5)^5 \times E]$. This set is closed since both $\mathbb{P}^5)^5$ and E are closed sets. $V \cap W$ is the set $\{(Q_1, Q_2, Q_3, Q_4, Q_5, P_s)\} : P_s \in \bigcap_{i=1}^{5} \tilde{T}_{Q_i}$ where P_s is described by the coefficients of a double line conic, and a crossed line conic. $V \cap W$ is a closed set since both V and W are closed and the projection from $V \cap W$ onto $(\mathbb{P}^5)^5$ is therefore closed.

Thus the set $\{Q_1, Q_2, Q_3, Q_4, Q_5\}$ such that there exists some $P \in \bigcap_{i=1}^{5} \tilde{T}_{Q_i}$ where P is described by the coefficients of a singular conic, is closed in $(\mathbb{P}^5)^5$. We now must show that its open complement is non-empty.

□

The proofs of both Lemma 38 and 39 depend on the existence of some arrangement of five conics that give rise to 3264 non-singular conics of multiplicity one. Fortunately the mathematicians Ronga, Tognoli, and Vust have constructed such an arrangement in their paper *The number of conics tangent to five given conics: the real case*[RTV].

Lemma 38 and Lemma 39 establish the general case for the solution set of conics in Steiner's problem. An identical proof can be used to establish similar results for all variations of Steiner's Problem involving points, lines and conics. One just needs to be careful to work with the appropriate varieties in $\mathbb{P}^5$.

7.4 General Position

During the course of the final calculations that yield the solutions to the various problems, we will introduce the appropriate definition of *general position* for a particular set of problems. It is not sufficient merely to say that a point, line, conic set is in general position if it gives rise to a set of conics that have no multiplicity. The conditions in $\mathbb{P}^2$ must also give rise to sets of curves that are composed of *non-singular* conics. We now take a look one specific problem, namely Steiner's Problem, and examine how general position fits into this example. This will help the reader to understand the motivation behind the definitions for general position given in the following section.

Consider five conics $C_1, C_2, C_3, C_4, C_5 \in \mathbb{P}^2$. We know that for each C_i there is a corresponding hypersurface $T_{C_i} \subset \mathbb{P}^5$. Let $\tilde{T}_{C_i} \subset Bl_V(\mathbb{P}^5)$ be the the pre-image of T_{C_i} under our defined blowup mapping. Let E denote the exceptional divisor as usual. If the set of conics tangent to the five fixed conics are non-singular, then we know that $\{\bigcap_{i=1}^{5} \tilde{T}_{C_i}\} \cap E = \emptyset$.

Consider a point P that lies on the intersection of $\{\bigcap_{i=1}^{5} \tilde{T}_{C_i}\}$ and the exceptional divisor, then we know that its coordinates can be thought of as the coefficients of a double line together with the coefficients of some pair of crossed lines. The double line conic giving the first factor of P is tangent to each C_i in $\mathbb{P}^2$. Also, the pair of crossed line conics giving the second factor of P, must be tangent to each $\check{C}_i$ in $\check{\mathbb{P}}^2$.

Now we know that every double line conic is tangent to each C_i, so our problem is reduced to finding configurations of five conics such that a fixed pair of crossed lines is tangent to each plane curve. We can then take the dual of the configuration to decern proper locations for each of the five fixed conics in Steiner's Problem.

7.5 Calculations

Definition 40. A configuration of points and lines in the plane is in *special position* if:

1. Three or more points are co-linear.

2. Three or more lines pass through a single point.

3. One or more points lie on a given line.

   The configuration is in *general position* if it is not in special position.

Theorem 41. *Given any two points $P_1, P_2 \in \mathbb{P}^2$ and any three lines $L_1, L_2, L_3 \subset \mathbb{P}^2$ such that the points and lines are in general position, there exist four conics that pass through the set of fixed points and are tangent to the fixed lines.*

Proof. We begin by considering the *dual* of our fixed point-line configuration. Thus we have three points and two lines in $\check{\mathbb{P}}^2$. By Theorem 16 we know that there are four conics in

$\check{\mathbb{P}}^2$ that satisfy these conditions. Invoking Lemma 19 we see that there are four conics that satisfy our given constraints. □

Theorem 42. *Given four lines and one point in $\mathbb{P}^2$ in general position, there are two conics that pass through the point and are tangent to the lines.*

Proof. Considering the *dual* of our given configuration. We have four points and one line fixed in $\check{\mathbb{P}}^2$. Theorem 2 established that there are two conics that pass through these points and are tangent to the line. Hence by Lemma 19 there are two conics in $\mathbb{P}^2$ that pass through one point and are tangent to four lines. □

Definition 43. A configuration of points, lines, and conics in the plane is in special position if:

1. Three or more points are co-linear.
2. Three or more lines pass through a single point. A point lies on a given line or a given conic. Three or more conics intersect each other in a single point.
3. A given conic and line are tangent to one another.
4. A conic is tangent to the line defined by the intersection points of two other conics.
5. Three or more conics have a common tangent line.

The configuration is in *general position* if it is not in special position.

Theorem 44. *Given four points $P_0, P_1, P_2, P_3 \in \mathbb{P}^2$ and a conic $C_0 \subset \mathbb{P}^2$ in general position, there are six conics that are tangent to C_0 and pass through the four given points.*

Proof. The statement is equivalent to the evaluation of $(T_Q)(T_P)^4$.

$$(T_Q)(T_P)^4 = (2T_P + 2T_L)(T_P)^4 = 2T_L{T_P}^4 + 2{T_P}^5 = 6.$$

□

Theorem 45. *Given four lines and a conic C_0 in general position there are six conics that are tangent to both C_0 and the four given lines.*

Proof. This follows from 44 and duality. $\square$

Theorem 46. *Given three points and two conics $C_0, C_1 \subset \mathbb{P}^2$ in general position, there are 36 conics that are tangent to C_0 and C_1 and pass through the three given points.*

Proof. The statement is equivalent to the evaluation of $(T_Q)^2(T_P)^3$.

$$(T_Q)^2(T_P)^3 = (2T_P + 2T_L)^2(T_P)^3 = 4{T_L}^2{T_P}^3 + 8T_L{T_P}^4 + 4{T_P}^5 = 36.$$

$\square$

Theorem 47. *Given three lines $L_0, L_1, L_2 \in \mathbb{P}^2$ and two conics $C_0, C_1 \subset \mathbb{P}^2$ in general position, there are 36 conics that are tangent to C_0 and C_1 and the given lines.*

Proof. This follows from theorem 46 and duality. $\square$

Theorem 48. *Given three points one line and one conic in general position, there are 12 conics that pass through the given points and are tangent to both the given line and conic.*

Proof. The statement is equivalent to the evaluation of $(T_L)(T_Q)(T_P)^3$.

$$(T_L)(2T_P + 2T_L)(T_P)^3 = 2{T_L}^2{T_P}^3 + 2T_L{T_P}^4$$

$\square$

Theorem 49. *Given three lines one point and one conic in general position, there are 12 conics that pass through the given point and are tangent to both the given lines and conic.*

Proof. The proof follows directly from the principal of duality and Theorem 48 $\square$

Theorem 50. *Given two points two lines and one conic in general position, there are 16 conics that pass through the two points and are tangent to the two lines and one conics.*

Proof. The statement is equivalent to the evaluation of $(T_L)^2(T_Q)(T_P)^2$.

$$(T_L)^2(2T_P + 2T_L)(T_P)^2 = 2{T_L}^3{T_P}^2 + 2{T_L}^2{T_P}^3 = 16.$$

□

Theorem 51. *There are 184 conics tangent to 3 fixed conics and tangent to 2 fixed lines in general position.*

Proof. The statement is equivalent to the evaluation of $(T_Q)^3T_L^2$.

$$[2T_P + 2T_L]^3T_L^2 = 8T_P^3T_L^2 + 24T_P^2T_L^3 + 24T_PT_L^4 + 8T_L^5$$

$$(T_Q)^3T_L^2 = 8(4) + 24(4) + 24(2) + 8(1) = 184$$

□

Theorem 52. *There are 184 conics tangent to 3 fixed conics and passing through 2 fixed points such that the points and conics are in general position.*

Proof. The result follows from Theorem 51 by duality. □

Theorem 53. *There are 880 conics tangent to 4 fixed conics and tangent to 1 fixed line such that the line and conics are in general position.*

Proof. The statement is equivalent to the evaluation of $(T_Q)^4T_L$.

$$[2T_P + 2T_L]^4T_L = 16T_P^4T_L + 64T_P^3T_L^2 + 96T_P^2T_L^3 + 64T_PT_L^4 + 16T_L^5$$

$$(T_Q)^4T_L = 16(2) + 64(4) + 96(4) + 64(2) + 16(5) = 880$$

□

Theorem 54. *There are 880 conics tangent to 4 fixed conics and passing through 1 fixed point where the conics and point are in general position.*

Proof. This follows from Theorem 53 by duality. □

Theorem 55. *There are 224 conics passing through one point tangent to one line and tangent to three conics where the point, line and conic are in general position.*

Proof. The statement is equivalent to the evaluation of $T_Q^3 T_P T_L$.

$$T_Q^3 T_P T_L = 8{T_L}^4 T_P + 24{T_L}^3{T_P}^2 + 24{T_L}^2{T_P}^3 + 8T_L{T_P}^4$$

$$T_Q^3 T_P T_L = 8(1) + 24(4) + 24(4) + 8(1) = 224.$$

□

Theorem 56. *There are 56 conics passing through two points tangent to one line and tangent to two conics in general position.*

Proof. The statement is equivalent to the evaluation of ${T_P}^2 T_L {T_Q}^2$.

$${T_P}^2 T_L {T_Q}^2 = 4{T_L}^3 T{P_2}^2 + 8{T_L}^2 TP_3 + 4T_L T{P_2}^4$$

$$4(4) + 8(4) + 4(2) = 56.$$

□

Theorem 57. *There are 56 conics passing through one points tangent to two lines and tangent to two conics in general position.*

Proof. This follows from theorem 56 and duality. □

We can utilize this method to settle the famous *Steiner Problem.*

Theorem 58. *There are 3264 plane conics that are tangent to five fixed conics that are arranged in the plane in general position.*

Proof. We need to compute $(\tilde{T_Q})^5$. We know that $(\tilde{T_Q})^5 = [2\tilde{T_P} + 2\tilde{T_L}]^5$ and

$$[2\tilde{T_P} + 2\tilde{T_L}]^5 = 32(\tilde{T_P}^5 + 5\tilde{T_P}^4\tilde{T_L} + 10\tilde{T_P}^3\tilde{T_L}^2 + 10\tilde{T_P}^2\tilde{T_L}^3 + 5\tilde{T_P}\tilde{T_L}^4 + \tilde{T_L}^5)$$

$$(\tilde{T_Q})^5 = 32(1 + 5(2) + 10(4) + 10(4) + 5(2) + 1) = 3264.$$

□

Table 1 lists the number of conics passing through **P** points, tangent to **L** lines, and tangent to **C** conics. We can see the glaring discrepancy between the expected number of conics one would expect from naively applying Bezout's Theorem and the actual number of conics that satisfy the given conditions.

P	**L**	**C**	**Expected Number**	**Actual Number**
5	0	0	1	1
4	1	0	2	2
3	2	0	4	4
2	3	0	8	4
1	4	0	16	2
0	5	0	32	1
4	0	1	6	6
3	0	2	36	36
2	0	3	216	184
1	0	4	1296	880
0	4	1	96	6
0	3	2	288	36
0	2	3	864	184
0	1	4	2592	880
1	1	3	432	224
2	1	2	72	56
3	1	1	12	12
1	3	1	48	12
2	2	1	24	16
1	2	2	144	56
0	0	5	7776	3264

Table 1: Solutions to Variations on Steiner's Problem

8 Application to a Classical Problem

A famous problem in classical geometry is Appolonius' Problem. Appolonius of Perga was a Greek geometer who studied at Euclid's School in Alexandria. His seminal work *Conics* is considered one of the greatest mathematical books of all time.

Appolonius' Problem is concerned with the possibility of constructing circles that are tangent to three fixed geometric structures in the plane. For example, a version of the problem asks for the construction of circles that passes through a fixed point, are tangent to a fixed line, and are tangent to a fixed circle. The most difficult variation is know as the Appolonius circle problem. In this problem, three circles are drawn in the plane and the challenge is to construct all circles that are tangent to the given circles. This was solved in the sixteenth century by the French mathematician Francois Viete. He was able to construct, with a straight edge and compass, eight circles that are all tangent to the three given circles. The constructed curves have become known as *Appolonius Circles* [EW].

We can make use of some modern methods in Algebraic Geometry and revisit the famous circle problem. Suppose we are not interested in actually constructing circles, but only wish to count the number of Appolonius Circles that exist in the complex projective plane. By working within the more general projective space, as well as over the field of complex numbers we will have a complete view of all possible Appolonius Circles.

Circles are easily visualized in Euclidean space where there is a well defined notion of distance. We understand circles to be the set of all points that are equidistant from a fixed point. A slightly different definition is needed when working with circles in the projective plane.

Definition 59. A *circle* is a conic that passes through the points $[1 : i : 0]$ and $[1 : -i : 0]$ in $\mathbb{P}^2$ [F].

For example, begin with the affine plane circle $x^2 + y^2 = 1$ and homogenize to obtain the defining equation $x^2 + y^2 = z^2$. We see that this meets the line at infinity ($z = 0$) at the two points $[1 : i : 0]$ and $[1 : -i : 0]$. $\mathbb{P}^5$ is the natural moduli space when working with general

conics of the form $ax^2 + by^2 + cz^2 + dxy + exz + fyz = 0$. From our above definition we see that circles are special conics that pass through two fixed points on the line at infinity. By constraining general conics and forcing them to pass through two points in $\mathbb{P}^2$ we form two linearly independent hyperplanes in $\mathbb{P}^5$. The intersection of these codimension-one linear surfaces is the three dimensional moduli space of all circles.

We can now derive the general equation for a projective circle. Consider the general equation for a conic

$$ax^2 + by^2 + cz^2 + dxy + exz + fyz = 0.$$

When we force this conic to pass through our two special points at infinity we arrive the two linear conditions

$$a - b + di = 0$$

and

$$a - b - di = 0.$$

From here we see that a must equal b and $d = 0$. Thus projective circles are of the form

$$ax^2 + ay^2 + cz^2 + exz + fyz = 0.$$

We parameterize the circle $Q : x^2 + y^2 = z^2$ with the following function:

$$\mathbb{P}^1 \longrightarrow \mathbb{P}^2$$

$$[s : t] \longmapsto [\frac{t^2 - s^2}{2} : st : \frac{s^2 + t^2}{2}].$$

Restricting the set of all circles to Q by evaluating our general equation at the image of the above function we obtain:

$$(-1/4)(s^2 + t^2)(-s^2c - s^2a + s^2e - 2fst - ct^2 - at^2 - t^2e) = 0.$$

Immediately we see that the expression vanishes when (s^2+t^2) vanishes, ie. when $s = \pm it$. This accounts for the fact that the line at infinity intersects all projective circles at the points $[1 : i : 0]$ and $[1 : -i : 0]$.

If we assume that $s \neq 0$, then we can dehomogenize the quantity on the right to a new variable $k = t/s$ and derive the the expression

$$(-a - c - e)k^2 - 2fk + (e - c - a) = 0.$$

The discriminant of this equation vanishes along a subvariety that corresponds to the set of all circles that are tangent to our fixed circle Q. This is a degree 2 hypersurface in $\mathbb{P}^3$ given by the equation

$$\tilde{T_Q} : 4f^2 - 4(-a - c - e)(e - c - a) = 0.$$

This hypersurface $\tilde{T_Q}$ is rationally equivalent to $2H$ for some codimension-one linear subvariety H. Evaluating $(\tilde{T_Q})^3$ will generate the moduli space of circles that are simultaneously tangent to three fixed circles. Since we are utilizing $\mathbb{P}^3$ as our moduli space of all circles, the transverse intersection of three hypersurfaces reduces to a finite number of points. Thus

$$(\tilde{T_Q})^3 = (2H)^3 = 8.$$

This verifies the solution to the classical problem proposed by Appolonius.

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